THERMAL STRESSES OF A THICK CIRCULAR PLATE DUE TO HEAT GENERATION

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ABSTRACT
In this paper, an attempt has been made to study thermoelastic response of a thick circular plate occupying the space \( D: 0 \leq r \leq b, -h \leq z \leq h \), due to heat generation with radiation type boundary conditions. We apply integral transform technique to find the thermoelastic solution.

AMS Subject Classification: 74A25, 74H39, 74D99.
Keywords: Thermoelastic Response, Circular Plate, Thermal Stresses.

1. INTRODUCTION
The direct and inverse problems of thermoelasticity of thick circular plate are considered by Nowacki. The quasi-static thermal stresses in circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature has determined by Wankhede. Noda et al. has succeeded in determining the quasi-static thermal stresses in a circular plate subjected to transient temperature along the circumference of circular upper face with lower face at zero temperature and the fixed circular edge thermally insulated.

In all aforementioned investigation they have not considered any thermoelastic problem with radiation type boundary conditions. This paper is concerned with transient thermoelastic problem of a thick circular plate occupying the space \( D: 0 \leq r \leq b, -h \leq z \leq h \), due to heat generation with radiation type boundary conditions.

2. STATEMENT OF THE PROBLEM
Consider thick circular plate of thickness \( 2h \) occupying the space \( D: 0 \leq r \leq b, -h \leq z \leq h \), the material is homogenous and isotropic. The differential equation governing the displacement potential function \( \phi(r,z,t) \) is given by
\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1+\nu}{1-\nu} \alpha T
\]
(1)

Where \( \nu \) and \( \alpha \) are Poisson’s ratio and linear coefficient of thermal expansion of the material of the plate and \( T \) is the temperature of the plate satisfying the differential equation
\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r,z,t) = \frac{1}{k} \frac{\partial T}{\partial t}
\]
(2)

Subject to initial condition
\[
M_1(T,1,0,0) = F(r,z) \quad 0 \leq r \leq b, -h \leq z \leq h.
\]
(3)

And boundary conditions are
\[
M_r(T,1,0,0) = g_1(z, t), \quad -h \leq z \leq h, \ t > 0
\]
(4)

\[
M_z(T,1,1,k_1,h) = f_1(r, t) \quad 0 \leq r \leq b, \ t > 0
\]
(5)

Where \( k \) is thermal diffusivity of material of the plate.

The displacement function in the cylindrical coordinate system are represented by Michell’s function as
\[
u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \quad \text{and} \quad u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}
\]
(6)
The Michell’s function must satisfy \( \nabla^2 \nabla^2 M = 0 \) (7)

where \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \)

The component of stresses are represented by the thermoelastic displacement potential \( \phi \) and Michell’s function \( M \) as

\[
\sigma_{rr} = 2G \left\{ \frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi \right\} + \frac{\partial}{\partial z} \left( \nu \nabla^2 M - \frac{\partial^2 M}{\partial r^2} \right) 
\]

\[
\sigma_{\theta \theta} = 2G \left\{ \frac{1}{r} \frac{\partial \phi}{\partial r} - \nabla^2 \phi \right\} + \frac{\partial}{\partial z} \left( \nu \nabla^2 M - \frac{1}{r} \frac{\partial^2 M}{\partial r^2} \right) 
\]

\[
\sigma_{zz} = 2G \left\{ \frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi \right\} + \frac{\partial}{\partial z} \left( (1-\nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) 
\]

\[
\sigma_{rz} = 2G \left\{ \frac{\partial \phi}{\partial z} + \frac{\partial}{\partial r} \left( (1-\nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right\} 
\]

For traction free surface stress function
\( \sigma_z = \sigma_{\theta z} = 0 \) at \( z = \pm h \) for thick plate.

Equation (1) to (11) constitute the mathematical formulation of the problem under consideration.

**3. THE SOLUTION OF THE PROBLEM**

Applying Hankel transform to the equation (2), we get

\[
-\xi_m^2 T^* (\xi_m, z, t) + \frac{d^2 T^*}{dz^2} (\xi_m, z, t) + T^* (\xi_m, z, t) = \frac{1}{k} \frac{dT^*}{dt}
\]

Again applying March-Fasulo transform to above equation, we obtain

\[
\frac{dT^*}{dt} + kP^2 T^* = \phi^* + \chi_1^* 
\]

\[
P^2 = \xi_m^2 + a_n^2, \phi_1^* = k\phi^*, \chi_1^* = k\chi_F
\]

Where

Equation (12) is a linear equation whose solution is given by

\[
T^* (\xi_m, n, t) = e^{-kP^2 t} \left[ \phi_1^* + \chi_1^* \right] e^{kp^2 t} dt^1 + Ce^{-kP^2 t}
\]

Using (3) we get

\[
C = F^* (m, n)
\]

Thus we have

\[
\bar{T}^* (\xi_m, n, t) = e^{-kP^2 t} \left[ \phi_1^* + \chi_1^* \right] e^{kp^2 t} dt^1 + \bar{F}^* (m, n)
\]

Applying inversion of Marchi-Fasulo transform and Hankel transform to the differential equation (13) one obtains

\[
T(r, z, t) = \frac{2}{b^2} \sum_n \sum_m J_0 \left( r \xi_m \right) P_n (z) \frac{P_n (z)}{J_1 (b \xi_m)} \int_0^t \left[ \phi_1^* + \chi_1^* \right] e^{kp^2 t} dt^1 + \bar{F}^* (m, n)
\]

This is the desired solution of the given problem.

Let us assume Michell’s function \( M \), which satisfy condition (10) as
\[ M(r, z) = \frac{2}{b^2} \sum_{m} \sum_{n} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi \]

\[ \psi = e^{-i\psi i} \left[ \int_0^t \left( \psi^* + \chi \right) e^{i\psi i} dt^1 + \tilde{F}^* (m,n) \right] \]

Where

\[ \phi \]

To obtain displacement potential using (1) and (14) we get

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \left( \frac{1+\nu}{1-\nu} \right) \alpha_t \frac{2}{b^2} \sum_{m} \sum_{n} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi \]

Solving above equation, we get

\[ \phi_1 = \left( \frac{1+\nu}{1-\nu} \right) \alpha_t \frac{2}{b^2} \sum_{m} \sum_{n} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi \]

\[ \phi_2 = \left( \frac{1+\nu}{1-\nu} \right) \alpha_t \frac{2}{b^2} \sum_{m} \sum_{n} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} B(t) \]

From equation (16) and (17), we get

\[ \phi = \phi_1 + \phi_2 = A \sum \sum \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi + B(t) \]

Where

\[ A = \left( \frac{1+\nu}{1-\nu} \right) \alpha_t \frac{2}{b^2} \int e^{-i\psi t} \int \left( \psi^* + \chi \right) e^{i\psi t} dt^1 + \tilde{F}^* (m,n) dt \]

4. DETERMINATION OF THERMAL DISPLACEMENT:

Substituting equations (15) and (18) in equation (6), we get

\[ u_{\gamma} = A \sum \sum \frac{\xi}{m} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi + B(t) - \frac{2}{b^2} \sum \sum \frac{\xi}{m} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi \]

and

\[ u_{\zeta} = \sum \sum \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi + B(t) \]

+ 2(1-\nu) \left[ \frac{2}{b^2} \sum \sum \frac{\xi}{m} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi + \frac{2}{b^2} \sum \sum \frac{\xi}{m} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi \right] \]

Substituting equations (15) and (18) in equations (8) to (11), we get

\[ \sigma_{rr} = 2G \left[ \frac{2(\nu-1)}{b^2} \sum \sum \frac{\xi}{m} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi + 2 \frac{r}{b^2} \sum \sum \frac{\xi}{m} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi \right] \]

\[ + \frac{2}{b^2} \sum \sum \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi - A \sum \sum \frac{\xi}{m} \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi + B(t) \]

\[ - A \sum \sum \frac{J_0(r \frac{\xi}{m})}{J_1(b \frac{\xi}{m})^2} \frac{P_n(z)}{\lambda_n} \psi + B(t) \]
\[
\sigma_{\text{bo}} = 2G \left\{ \frac{2V}{b^2} \sum_{n=1}^{\infty} \frac{\xi_n J_1(\xi_n r)}{\lambda_n} - \frac{2(V-1)}{b^2} \sum_{n=1}^{\infty} \frac{\xi_n J_1(\xi_n r) P_n^1(r)}{\lambda_n} \psi + \frac{2V}{b^2} \sum_{n=1}^{\infty} \frac{\xi_n J_1(\xi_n r) P_n^1(r)}{\lambda_n} \psi + B(t) \right\}
\]

\[
\sigma_{zz} = 2G \left\{ \frac{2(1-V)}{b^2} \sum_{n=1}^{\infty} \frac{\xi_n J_1(\xi_n r) P_n^1(r)}{\lambda_n} - \frac{2(1-V)}{b^2} \sum_{n=1}^{\infty} \frac{\xi_n J_1(\xi_n r) P_n^1(r)}{\lambda_n} \psi + \frac{2(1-V)}{b^2} \sum_{n=1}^{\infty} \frac{\xi_n J_1(\xi_n r) P_n^1(r)}{\lambda_n} \psi + B(t) \right\}
\]

\[
\sigma_{z\theta} = 2G \left\{ \frac{2(1-V)}{b^2} \sum_{n=1}^{\infty} \frac{\xi_n J_1(\xi_n r) P_n^1(r)}{\lambda_n} - \frac{2(1-V)}{b^2} \sum_{n=1}^{\infty} \frac{\xi_n J_1(\xi_n r) P_n^1(r)}{\lambda_n} \psi + \frac{2(1-V)}{b^2} \sum_{n=1}^{\infty} \frac{\xi_n J_1(\xi_n r) P_n^1(r)}{\lambda_n} \psi + B(t) \right\}
\]

Where
\[
A = \left( \frac{1+V}{1-V} \right) \frac{2\alpha}{b^2}, \psi = e^{-ib\cdot} \int_0^1 \left( \psi^* + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi \right) dt + B(t) = \int \psi dt
\]

5. SPECIAL CASE:
Set \( F(r, z) = z^2 (1 - r^2) \)
Applying Marchi-Fasulo transform, are obtain
\[
\overline{F}(r, n) = (1 - r^2) \int_0^h \int \overline{F}(r, z) dz
\]

\[
\overline{F}(r, n) = (1 - r^2) \phi_n \left[ \frac{2h^2 \sin(a_n h)}{a_n} + 4h \cos(a_n h) - 4 \sin(a_n h) \right]
\]

Where \( P_n(z) = \phi_n \cos(a_n z) - w_n \sin(a_n z) \)

\[
\phi_n = a_n (\alpha_1 + \alpha_2) \cos(a_n h) + (\beta_1 - \beta_2) \sin(a_n h)
\]

\[
w_n = (\beta_1 - \beta_2) \cos(a_n h) + a_n (\alpha_1 - \alpha_2) \sin(a_n h)
\]

And
Again on applying Hankel transform, we obtain
\[
\overline{F}^* (m, n) = D_m \left[ \frac{b}{\xi_m} J_1(\xi_m) - \frac{b^2 \xi_m^2 - 4}{\xi_m} J_0(\xi_m) \right] - \frac{2b^2}{\xi_m^2} J_0(\xi_m)
\]

Where
\[
D_m = \phi_m \left[ \frac{2h^2 \sin(a_n h)}{a_n} + 4h \cos(a_n h) - 4 \sin(a_n h) \right]
\]

And

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Using equation (25) in equation (14), one obtains

\[ T(r,z,t) = \frac{2}{b^2} \sum_{m} \sum_{n} \frac{J_0(r \xi_m)}{J_1(b \xi_m)} e^{-k \xi_m^2 t} \left( \int_0^t \left( e^\lambda \right) e^{k \xi_m^2 t} dt \right) D_0 \left( \frac{b}{\xi_m} J_1(b \xi_m) - \frac{b(b^2 \xi_m^2 - 4)}{\xi_m^3} J_1(b \xi_m) - 2 \frac{b^2}{\xi_m^2} J_0(b \xi_m) \right) \]

(26)

6. NUMERICAL RESULTS:

Set \( b = 2, k = 15.9 \times 10^6 \) s^{-1} in equation (26), we get

\[ T(r,z,t) = \frac{2}{4} \sum_{m} \sum_{n} \frac{J_0(r \xi_m)}{J_1(b \xi_m)} e^{-k \xi_m^2 t} \left( \int_0^t \left( e^\lambda \right) e^{k \xi_m^2 t} dt \right) D_0 \left( \frac{2}{\xi_m} J_1(2 \xi_m) - \frac{2(4 \xi_m^2 - 4)}{\xi_m^3} J_1(2 \xi_m) - \frac{2}{\xi_m^2} J_0(2 \xi_m) \right) \]

(27)

7. CONCLUSION

The temperature distribution, displacement and thermal stresses of thick circular plate are investigated with known boundary conditions. Finite integral transform techniques are used to obtain numerical results. Any particular cases of special interest can be assigned to the parameters and functions in expressions. The temperature, displacement and thermal stresses that are obtained can be useful to the design of structure or machines in engineering applications.

Reference