THERMOELASTIC PROBLEM OF A THIN RECTANGULAR PLATE DUE TO PARTIALLY DISTRIBUTED HEAT SUPPLY

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ABSTRACT

In this paper, an attempt has been made to solve three dimensional transient thermoelastic problem of a thin rectangular plate due to partially distributed heat supply to determine temperature distribution, displacement and thermal stresses with the known boundary and initial conditions by applying the Marchi-Fasulo transform and the Laplace transform techniques.

Keywords: Thin rectangular plate, three dimensional transient thermoelastic problem, integral transform

1 INTRODUCTION

From last hundreds years, lots of analytical approach for finding solution to the plane problems in terms of stresses are derived. The primary objective of the present paper is to gain an effective solution and a better understanding of thermal stresses in thin rectangular plate due to partially distributed heat supply. Tanigawa and Komatsubara [1997], Vihak et al [1998] and Adams and Bert [1999] have studied the direct problem of thermoelasticity in a rectangular plate under thermal shock. Khobragade [2003] have studied the inverse steady state thermoelastic problem to determine the temperature displacement function and thermal stresses at the boundary of a thin rectangular plate. They have used the finite Fourier sine transform technique. Nowacki [1957] has determined the steady-state thermal stresses in a circular plate subjected to an ax symmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge respectively.

Quasi-static thermal stresses in a thin circular plate subjected to transient temperature along the circumference of circular upper face with lower face is at zero temperature and the fixed circular edge thermally insulated determined by Roychoudhary [1973]. Transient thermoelastic-plastic bending problems of a circular plate has discussed by Ishihara [1997]. A method of direct integration for determination of stresses and displacements within the framework of elasticity and thermoelasticity problems in terms of stresses proposed by Vihak et al [1995, 1998, 1999, 2000]. This method was developed with respect to the construction of

solutions to the problems for annular domains in addition to the construction of expressions for stresses, the method of direct integration gives a lot of advantages, among which is the determination of the integral conditions of equilibrium and compatibility. The mentioned conditions play an important role in both the determination of the stress-strain state and the verification of numerical results.

To the author’s knowledge, work on three dimensional inverse transient thermoelastic problem of a thin rectangular plate with given third kind boundary conditions has not been yet reported.

In the present paper, an attempt has been made to determine the temperature, displacement and thermal stress at any point of a thin rectangular plate occupying the region $D: x \in (-a,a), y \in (-b,b), z \in (0,h)$, with known boundary conditions. Here the Marchi-Fasulo transform and the Laplace transform techniques have been used to find the solution of the problem.

2 STATEMENT OF THE PROBLEM

Consider a thin rectangular plate occupying the space $D: x \in (-a,a), y \in (-b,b), z \in (0,h)$. The displacement components $u_x, u_y, u_z$ in the $x$, $y$ and $z$ direction respectively are in the integral form as

$$u_x = \int_a^b \left[ 1 \left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx$$

$$u_y = \int_b^a \left[ 1 \left( \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy$$

$$u_z = \int_c^d \left[ 1 \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial z^2} \right) + \alpha T \right] dz$$

Where $E, \nu$ and $\alpha$ are the Young modulus, the Poisson ratio and the linear coefficient of thermal expansion of the material of the plate respectively, $U(x,y,z,t)$ is the Airy stress function which satisfies the differential equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)^2 U(x,y,z,t) = -\alpha \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)^2 T(x,y,z,t)$$

Here $T(x,y,z,t)$ denotes the temperature of thin rectangular plate satisfying the following differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k} \frac{\partial T}{\partial t}$$

and $k$ is the thermal diffusivity of the material subject to initial conditions.

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\[ T(x, y, z, t) = 0 \]  
\quad (2.6)

The boundary conditions are

\[ \left[ T(x, y, z, t) + k_1 \frac{\partial T(x, y, z, t)}{\partial x} \right]_{x=a} = F_1(y, z, t) \]  
\quad (2.7)

\[ \left[ T(x, y, z, t) + k_2 \frac{\partial T(x, y, z, t)}{\partial x} \right]_{x=b} = F_2(y, z, t) \]  
\quad (2.8)

\[ \left[ T(x, y, z, t) + k_3 \frac{\partial T(x, y, z, t)}{\partial y} \right]_{y=a} = F_3(x, z, t) \]  
\quad (2.9)

\[ \left[ T(x, y, z, t) + k_4 \frac{\partial T(x, y, z, t)}{\partial y} \right]_{y=b} = F_4(x, z, t) \]  
\quad (2.10)

\[ \left[ T(x, y, z, t) + c \frac{\partial T(x, y, z, t)}{\partial z} \right]_{z=0} = -\frac{Q}{\lambda} f(x, y, t) \]  
\quad (2.11)

\[ \left[ T(x, y, z, t) + c \frac{\partial T(x, y, z, t)}{\partial z} \right]_{z=h} = g(x, y, t) \]  
\quad (2.12)

The stresses components in terms of \( U(x, y, z, t) \) are given by

\[ \sigma_{xx} = \left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) \]  
\quad (2.13)

\[ \sigma_{yy} = \left( \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} \right) \]  
\quad (2.14)

\[ \sigma_{zz} = \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \]  
\quad (2.15)

The equations (2.1) to (2.15) constitute the mathematical formulation of the problem under consideration.

3 SOLUTION OF THE PROBLEM

By applying the finite Marchi-Fasulo transform two times to equation (2.5) to (2.12), and then their inversion, we obtain
\[ T(x, y, z, t) = \frac{k}{c^2} \sum_{m,n} \left( \frac{P_m(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left[ \phi_1(z)t_1(t) \right] \]

\[ + \frac{2k \prod}{h^2} \sum_{m,n} \left( \frac{P_m(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left( \frac{l}{\cos l} \right) \left( \frac{1}{1 + \left( \frac{cl\pi}{h} \right)^2} \right) \left[ \eta_1(z)t_2(t) \right] \]

Where

\[ \phi_1(z) = \frac{\sinh \left( \frac{z}{c} \right) - \cos \left( \frac{z}{c} \right)}{\sinh \left( \frac{h}{c} \right)} \]

\[ \phi_2(z) = \frac{\sinh \left( \frac{z-h}{c} \right) - \cos \left( \frac{z-h}{c} \right)}{\sinh \left( \frac{h}{c} \right)} \]

\[ \eta_1(z) = \sin \left( \frac{l\pi}{h} \right) - \left( \frac{cl\pi}{h} \right) \cos \left( \frac{l\pi}{h} \right) \]

\[ \eta_2(z) = \left[ \sin \left( \frac{l\pi}{h} \right) - \left( \frac{cl\pi}{h} \right) \cos \left( \frac{l\pi}{h} \right) \right] (z-h) \]

\[ \psi_1(t) = \int_0^t f(m,n,t') e^{\left( \frac{1-c^2}{c^2} \right) t'} \, dt' \]

\[ \psi_2(t) = \int_0^t g(m,n,t') e^{\left( \frac{1-c^2}{c^2} \right) t'} \, dt' \]

\[ \psi_3(t) = \int_0^t f(m,n,t') e^{-k \left( \frac{l^2}{l^2} \right) t'} \, dt' \]

\[ \psi_4(t) = \int_0^t g(m,n,t') e^{-k \left( \frac{l^2}{l^2} \right) t'} \, dt' \]

Here \( f(m,n,t) \) and \( g(m,n,t) \) denote the Marchi-Fasulo transform of \( \tilde{f}(m,y,t) \) and \( \tilde{g}(m,y,t) \) respectively. \( \tilde{f}(m,y,t) \), \( \tilde{g}(m,y,t) \) denote the finite Marchi-Fasulo transform of \( f(x,y,t) \) and \( g(x,y,t) \) respectively.

\[ f(m, n, t) = \int_{-b}^{b} f(m, y, t) P_n(y) dy; \quad g(m, n, t) = \int_{-b}^{b} g(m, y, t) P_n(y) dy, \]

And

\[ \lambda_n = \int_{-b}^{b} P_n^2(y) dy \]

\[ P_n(y) = Q_n \cos(a_n y) - w_n \sin(a_n y), \]

\[ Q_n = a_n (\alpha_3 + \alpha_4) \cos(a, b) - (\beta_3 - \beta_1) \sin(a, b), \]

\[ w_n = a_n (\beta_3 + \beta_4) \cos(a, b) - (\alpha_4 - \alpha_3) \sin(a, b) \]

Equation (16) is the desired solution of the given problem with \( \beta_3 = \beta_4 = 1, \alpha_3 = k_3, \alpha_4 = k_4 \).

4 DETERMINATION OF THE AIRY STRESS FUNCTION

Substituting the values of \( T(x, y, z, t) \) from equation (3.1) in equation (2.4), one obtains

\[
U(x, y, z, t) = \frac{\alpha Ek}{c^2} \sum_{m,n=1}^{\infty} \left( \frac{P_m(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left[ \frac{i \phi_1(z) \psi_1(t) - i \phi_2(z) \psi_2(t)}{ a_m^2 + a_n^2 - \frac{1}{c^2} } \right] \\
+ \frac{2\alpha Ek \pi}{h^2} \sum_{l,m,n=1}^{\infty} \left( \frac{P_m(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \frac{l}{\cos \pi} \left[ \frac{i \eta_1(z) \psi_1(t) - i \eta_2(z) \psi_2(t)}{ a_m^2 + a_n^2 + \left( \frac{\pi}{h} \right)^2 } \right]
\]

(4.1)

5 DETERMINATION OF DISPLACEMENT COMPONENTS

Substituting the values of (4.1) in the equation (2.1) to (2.3), one obtains

\[
u_1 = \frac{\alpha k}{c^2} \sum_{m,n=1}^{\infty} \left( \frac{(k_1 + k_2) \sin 2a_n a}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left[ 1 + \nu \right] a_n^2 \left[ \frac{i \phi_1(z) \psi_1(t) - i \phi_2(z) \psi_2(t)}{ a_m^2 + a_n^2 - \frac{1}{c^2} } \right] \\
+ \frac{2\alpha \pi}{h^2} \sum_{l,m,n=1}^{\infty} \left( \frac{(k_1 + k_2) \sin 2a_n a}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \frac{l}{\cos \pi} \left[ \frac{(1 + \nu) a_m^2}{ a_m^2 + a_n^2 + \left( \frac{\pi}{h} \right)^2 } \right] \left[ \frac{i \eta_1(z) \psi_1(t) - i \eta_2(z) \psi_2(t)}{1 + \left( \frac{\pi}{h} \right)^2 } \right] 
\]

(5.1)
\[ u_y = \frac{\alpha k}{c^2} \sum_{m,n=1}^{\infty} \left( \frac{(k_3 + k_4) \sin 2\alpha_n}{\mu_n} \right) \frac{P_m(x)}{\lambda_m} \left( 1 + \nu \right) a_n^2 \left[ \phi_1(z) \psi_1(t) - \phi_2(z) \psi_2(t) \right] a_m^2 + a_n^2 - \frac{1}{c^2} \]

\[ + \frac{2\alpha k \pi}{h^2} \sum_{l,m,n=1}^{\infty} \left( \frac{(k_3 + k_4) \sin 2\alpha_n}{\mu_n} \right) \frac{P_m(x)}{\lambda_m} \left( 1 + \nu \right) a_n^2 \left[ \eta_1(z) \psi_1(t) - \eta_2(z) \psi_2(t) \right] \frac{1 + \left( \frac{l \pi}{h} \right)^2}{1 + \left( \frac{1 + \nu \alpha_m}{c^2} \right)} \]

\[ u_z = \frac{\alpha k}{c^2} \sum_{m,n=1}^{\infty} \left( \frac{P_m(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left( 1 + \nu \right) a_n^2 \left[ \phi_1'(h) \psi_1(t) - \phi_2'(h) \psi_2(t) \right] a_m^2 + a_n^2 - \frac{1}{c^2} \]

\[ + \frac{2\alpha k \pi}{h^2} \sum_{l,m,n=1}^{\infty} \left( \frac{P_m(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left( 1 + \nu \right) a_n^2 \left[ \eta_1'(h) \psi_1(t) - \eta_2'(h) \psi_2(t) \right] \frac{1 + \left( \frac{l \pi}{h} \right)^2}{1 + \left( \frac{1 + \nu \alpha_m}{c^2} \right)} \]

(5.2)

\[ \phi_1'(h) = \frac{\cosh \left( \frac{h}{c} \right) - \sin \left( \frac{h}{c} \right) - 1}{\frac{1}{\sinh \left( \frac{h}{c} \right)}} \]

\[ \phi_2'(h) = \frac{1 - \cosh \left( \frac{h}{c} \right) - \sin \left( \frac{h}{c} \right)}{\frac{1}{\sinh \left( \frac{h}{c} \right)}} \]

\[ \eta_1'(h) = \frac{-\cos \left( \frac{l \pi}{h} \right) - \left( \frac{l \pi}{h} \right) \sin \left( \frac{l \pi}{h} \right) h + 1}{\left( \frac{l \pi}{h} \right)} \]

\[ \eta_2'(h) = \frac{-\cos \left( \frac{l \pi}{h} \right) - \left( \frac{l \pi}{h} \right) \sin \left( \frac{l \pi}{h} \right) h + 1}{\left( \frac{l \pi}{h} \right)} \]

(5.3)

Where

\[ \phi_1'(h) = \frac{\cosh \left( \frac{h}{c} \right) - \sin \left( \frac{h}{c} \right) - 1}{\frac{1}{\sinh \left( \frac{h}{c} \right)}} \]

\[ \phi_2'(h) = \frac{1 - \cosh \left( \frac{h}{c} \right) - \sin \left( \frac{h}{c} \right)}{\frac{1}{\sinh \left( \frac{h}{c} \right)}} \]

\[ \eta_1'(h) = \frac{-\cos \left( \frac{l \pi}{h} \right) - \left( \frac{l \pi}{h} \right) \sin \left( \frac{l \pi}{h} \right) h + 1}{\left( \frac{l \pi}{h} \right)} \]

\[ \eta_2'(h) = \frac{-\cos \left( \frac{l \pi}{h} \right) - \left( \frac{l \pi}{h} \right) \sin \left( \frac{l \pi}{h} \right) h + 1}{\left( \frac{l \pi}{h} \right)} \]

6 DETERMINATION OF THE STRESS FUNCTION

Substituting the values of (4.1) in the equation (2.13) to (2.15), one obtains

\[
\sigma_{\alpha} = \frac{\alpha E_k}{c^2} \sum_{n,m=1}^{\infty} \left( \frac{P_n(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left[ \frac{1}{c^2} - a_n^2 \right] \left[ \phi_1(z)\psi_1(t) - \phi_2(z)\psi_2(t) \right] \frac{1}{a_n^2 + a_m^2 - \frac{1}{c^2}} \\
+ \frac{2\alpha E_k}{h^2} \sum_{n,m=1}^{\infty} \left( \frac{P_n(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left[ \frac{1}{c^2} - a_n^2 \right] \left[ \phi_1(z)\psi_1(t) - \phi_2(z)\psi_2(t) \right] \frac{1}{a_n^2 + a_m^2 - \frac{1}{c^2}} \\
\left( \frac{1}{\cos \pi \frac{h}{l}} - a_n^2 \right) \left[ \eta_1(z)\psi_1(t) - \eta_2(z)\psi_2(t) \right] \frac{1}{1 + \left( \frac{\pi}{h} \right)^2} \\
(6.1)
\]

\[
\sigma_{\beta} = \frac{\alpha E_k}{c^2} \sum_{n,m=1}^{\infty} \left( \frac{P_n(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left[ \frac{1}{c^2} - a_n^2 \right] \left[ \phi_1(z)\psi_1(t) - \phi_2(z)\psi_2(t) \right] \frac{1}{a_n^2 + a_m^2 - \frac{1}{c^2}} \\
+ \frac{2\alpha E_k}{h^2} \sum_{n,m=1}^{\infty} \left( \frac{P_n(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left[ \frac{1}{c^2} - a_n^2 \right] \left[ \phi_1(z)\psi_1(t) - \phi_2(z)\psi_2(t) \right] \frac{1}{a_n^2 + a_m^2 - \frac{1}{c^2}} \\
\left( \frac{1}{\cos \pi \frac{h}{l}} - a_n^2 \right) \left[ \eta_1(z)\psi_1(t) - \eta_2(z)\psi_2(t) \right] \frac{1}{1 + \left( \frac{\pi}{h} \right)^2} \\
(6.2)
\]

\[
\sigma_{\gamma} = \frac{\alpha E_k}{c^2} \sum_{n,m=1}^{\infty} \left( \frac{P_n(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left[ \frac{1}{c^2} - a_n^2 \right] \left[ \phi_1(z)\psi_1(t) - \phi_2(z)\psi_2(t) \right] \frac{1}{a_n^2 + a_m^2 - \frac{1}{c^2}} \\
+ \frac{2\alpha E_k}{h^2} \sum_{n,m=1}^{\infty} \left( \frac{P_n(x)}{\lambda_m} \right) \left( \frac{P_n(y)}{\mu_n} \right) \left[ \frac{1}{c^2} - a_n^2 \right] \left[ \phi_1(z)\psi_1(t) - \phi_2(z)\psi_2(t) \right] \frac{1}{a_n^2 + a_m^2 - \frac{1}{c^2}} \\
\left( \frac{1}{\cos \pi \frac{h}{l}} - a_n^2 \right) \left[ \eta_1(z)\psi_1(t) - \eta_2(z)\psi_2(t) \right] \frac{1}{1 + \left( \frac{\pi}{h} \right)^2} \\
(6.3)
\]

7 SPECIAL CASE AND NUMERICAL RESULTS

Setting,
\[
f(x, y, t) = (1 - e^{-t})(x + a)^2(x - a)^2(y + b)^2(y - b)^2,
\]
\[
g(x, y, t) = (1 - e^{-t})(x + a)^2(x - a)^2(y + b)^2(y - b)^2e^h,
\]
\[
\delta = \frac{8(k_1 + k_2)}{h^2}, \quad a = 0.75, \quad k = 0.86, \quad b = 0.75, \quad h = 0.25, \quad t = 1
\]
in the equation (3.1), we obtain
\[
T(x, y, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} (-1)^{l+1} \frac{1}{2} \left( \frac{P_n(y)}{\lambda_m} \right) \left[ \frac{P_n(x)}{\mu_n} \right] \phi(z)e^{-\psi(z)} \\
\times \left[ a_n^2 \cos^2(a_n) - \cos(a_n)\sin(a_n) \right] \\
(7.1)
\]

8 GRAPHICAL ANALYSIS

From the plotted graph between obtained temperatures versus t for different values of x, it is observed that as t vary from 1 to 3 seconds temperature decreases gradually and after time t=3 it becomes stable for different values of x, or we can say that as x increases, the temperature gradually decreases due to partially distributed heat supply.

The following graphs give the characteristic of stresses versus different values of t.
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Figure 3: $\sigma_{yy}$ versus $x$ for different values of $t$

Figure 4: $\sigma_{zz}$ versus $z$ for different values of $t$

Figure 5: $U(x, y, z, t)$ versus $t$ for different values of $z$
9 CONCLUSION

The temperature, displacements, and thermal stresses at any point of the plate have been obtained, when the boundary conditions are known with the aid of the finite Marchi-Fasulo transform technique. The expressions are represented graphically. The results are obtained in the form of infinite series. It is observed that as x increases, the temperature gradually decreases. Any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions.

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